LVI. *An Explication of an obſcure Paſſage in* Albert Girard's *Commentary upon* Simon Stevin's *Works (Vide* Les Oeuvres Mathem. de Simon Stevin, à Leyde, 1634, p. 169, 170); *by Mr.* Simſon, *Profeſſor of Mathematics at the Univerſity of* Glaſgow: *Communicated by the Right Honourable* Philip *Earl* Stanhope.

Read Dec. 20, 1753.

"PUIS que je ſuis entré en la matiere des nombres rationaux, "j'adjouſteray encore deux ou trois particularitez, "non encor par cy devant practiquées, comme d'expliquer les radicaux extremement pres, *&c.*"

The firſt thing Albert Girard gives in this place is a method of expreſſing the ratio of the ſegments of a line cut in extreme and mean proportion, by rational numbers, that converge to the true ratio. For this purpoſe he takes the progreſſion 0, 1, 1, 2, 3, 5, 8, 13, 21, *&c.* every term of which is equal to the ſum of the two terms that precede it: and ſays, any number in this progreſſion has unto the following the ſame ratio [nearly] that any other has to that, which follows it. Thus 5 has to 8 nearly the ſame ratio, that 8 has to 13; conſequently, any 3 numbers next one another as 8, 13, 21, nearly expreſs the ſegments of a line cut in extreme and mean proportion, and the whole line; ſo that 13, 21, V. *B.* 13 is wrong printed for the ſecond num ad of 21) conſtitute

ſtitute near enough an iſoſceles triangle, having the angle of a pentagon; *i. e.* whoſe angle at the vertex is ſubtended by the ſide of a pentagon in the circle deſcribed about the triangle.

Now this will be plain, if it be ſhewn, that the ſquares of the numbers in this ſeries are alternately leſſer and greater by an unit, than the product of the two numbers next them upon each ſide. Thus, in the four numbers, 5, 8, 13, 21, the ſquare of 8 is an unit leſſer than the product of 5 and 13; but the ſquare of 13 that next follows 8, *viz.* 169, is an unit greater than 8 times 21, or 168; and ſo on conſtantly.

Caſe 1.

If a, b, c, be ſuch numbers, that $\begin{cases} 1.\ a+b=c \\ 2.\ ac=bb+1 \end{cases}$

Then, if d be taken, ſo that $d=b+c$; then ſhall $bd+1=cc$.

Becauſe $d=b+c$; $bd+1$ ſhall be $=bb+bc+1=ac+bc$ [2] which is $=\overline{a+b}\times c=cc$ [1]: *Ergo* $bd+1=cc$.

Caſe 2.

If a, b, c, be ſuch that $\begin{cases} 1.\ a+b=c \\ 2.\ ac+1=bb \end{cases}$

Then, if d be taken, ſo that $d=b+c$; then ſhall $bd=cc+1$.

Becauſe $bd=bb+bc=ac+bc+1$ [2.] $=\overline{a+b}\times c+1$ $=cc+1$ [1.]

Problem.

Having given the number a, in *Caſe* 1. to find b and c, *i. e.* having given a to find b ſuch that $bb+1=(ac=)\ aa+ab$; then is $bb-ab=aa-1$:

and therefore $b = \frac{a + \sqrt{5aa - 4}}{2}$. Whence, to make b a rational integer number, $5aa - 4$ must be a square; which it will be, if $a = 1$; and then b will also be 1, and c will be 2: and having continued the series, every number will have the properties mentioned.

The second thing which Albert Girard mentions, is a way of exhibiting a series of rational fractions, that converge to the square root of any number proposed, and that very fast. He tells nothing about the way of forming it, and only gives the two following examples; *viz.*

He says, $\sqrt{2}$ is equal nearly to $\frac{577}{408}$: or, if you would have it nearer, to $\frac{1393}{985}$.

His other example is of $\sqrt{10}$, which, he says, is nearly equal to $3\frac{53353}{328776}$; *i. e.* to $\frac{1039681}{328776}$. And these are the fractions your lordship has turned, at first sight, into continued fractions of the same value *.

The way of making a series of rational fractions, which converge to the square root of any number proposed, in such a manner, that the square of the numerator of any of them being lessened by an unit, or, in some cases, increased by an unit, the remainder, or sum divided by the square of the denominator, shall be exactly equal to the number proposed, depends upon the following propositions:

Prop.

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* *N. B.* That the continued fraction here alluded to for expressing the square root of 10 was $\frac{1}{6} \times 19 - \cfrac{1}{38 - \cfrac{1}{38 - \cfrac{1}{38 - \cfrac{1}{38}}}}$, &c. *ad infinitum.*

Prop. 1.

Let a be any number proposed, and $\frac{b}{c}$ be such a fraction, that $\frac{bb-1}{cc}=a$, i. e. $bb=acc+1$, then, if two other fractions be taken, one of which is $\frac{b}{ac}$, the first divided by the proposed number a, and the other is $\frac{c}{b}$, the reciprocal of the first fraction; then the fraction $\frac{bb+acc}{2bc}$, whose numerator is the sum of the products of the numerators, and of the denominators of the fractions $\frac{b}{c}$ and $\frac{b}{ac}$; and its denominator the sum of the products of the numerators, and of the denominators of the fractions $\frac{b}{c}$ and $\frac{c}{b}$, shall have the same property with the fraction $\frac{b}{c}$ i. e. $\frac{\overline{bb+acc}^2-1}{2bc^2}=a$,

Because $bb=acc+1$

$$bb-acc=1, \text{ and squaring}$$

$$b^4-2ab^2c^2+a^2c^4=1. \text{ And adding } 4ab^2c^2$$

$$b^4+2ab^2c^2+a^2c^4=4ab^2c^2+1.$$

Whence $\frac{\overline{b^2+acc}^2-1}{2bc^2}=a$,

Prop. 2.

If $\frac{b}{c}$ be such a fraction, that $\frac{bb+1}{cc}=a$, i. e. $bb+1=acc$, all other things remaining as in *Prop.* 1.; then shall the fraction $\frac{bb+acc}{2bc}$, formed as there described, be such, that $\frac{\overline{bb+acc}^2-1}{2bc^2}=a$,

 Because

Beaufe $bb+1=acc$, then $acc-bb=1$; and fquaring

$$b^4-2ab^2c^2+a^2c^4=1.$$

Whence, as in the foregoing, it will follow, that $\frac{\overline{bb+acc}^2-1}{\overline{2bc}^2}=a.$

Prop. 3.

Let the fraction $\frac{b}{c}$ be fuch, that $\frac{bb-1}{cc}=a$, i. e. $bb=acc+1$; alfo let $\frac{d}{e}$ be another fraction, having the fame property with $\frac{b}{c}$, *i. e.* fuch, that $dd=aee+1$. Then, if from the fraction $\frac{d}{e}$, and the two others mentioned in *Prop.* 1. *viz.* $\frac{b}{ac}$, and $\frac{c}{b}$, a new fraction be formed, in the fame manner as the fraction $\frac{bb+acc}{2bc}$ was formed from $\frac{b}{c}$, and the fame two $\frac{b}{ac}$ and $\frac{c}{b}$, which fraction will be $\frac{bd+ace}{cd+be}$; this new fraction fhall have the fame property with the other two $\frac{b}{c}$ and $\frac{d}{e}$, *i.e.* $\frac{\overline{bd+ace}^2-1}{\overline{cd+be}^2}=a.$

Hypoth. 1. $bb=acc+1$

2. $dd=aee+1$

3. $ac^2d^2=a^2c^2e^2+ac^2$ [2.]

4. $b^2d^2=ab^2e^2+b^2$ [2.]

5. $b^2d^2=ab^2e^2+ac^2+1$ [4, 1.]

6. $b^2d^2+a^2c^2e^2=ab^2e^2+a^2c^2e^2+ac^2+1$ [5.]

7. $b^2d^2+a^2c^2e^2=ac^2d^2+ab^2e^2+1$ [6. 3]

8. $b^2d^2+2abcde+a^2c^2e^2=ac^2d^2+2abcde+ab^2e^2+1$ [7].

i. e.

i. e. $\overline{bd+ace}^2 = a \times \overline{cd+be}^2 + 1.$

9. $\dfrac{\overline{bd+ace}^2 - 1}{\overline{cd+be}^2} = a.$

Prop. 4.

The same things being supposed as in *Prop.* 3. except that bb, instead of being equal to $acc+1$, as there, is equal to $acc-1$, or $bb+1=acc$; it will follow, by the like steps as in *Proposition* 3. that $\dfrac{\overline{bd+ace}^2 + 1}{\overline{cd+be}^2} = a.$

Prop. 5.

If likewise d^2 be equal to $aee-1$, as well as $b^2 = acc-1$, all other things remaining as in *Proposition* 3. then shall $\overline{bd+ace}^2 = a \times \overline{cd+be}^2 + 1$, *i. e.* $\dfrac{\overline{bd+ace}^2 - 1}{\overline{cd+be}^2} = a.$

Hypoth. { 1. $b^2+1=acc$
2. $d^2+1=aee$

3. $b^2d^2+b^2=ab^2e^2$ [2.]
4. $ac^2d^2+ac^2=a^2c^2e^2$ [2.]
5. $b^2d^2+ac^2=ab^2e^2+1$ [3, 1.]
6. $b^2d^2+ac^2+ac^2d^2=ac^2d^2+a^2be^2+1$ [5.]
7. $b^2d^2+a^2c^2e^2=ac^2d^2+ab^2e^2+1$ [6.4.]
8. $b^2d^2+2abcde+a^2c^2e^2=ac^2d^2+2abcde+ab^2e^2+1$ [7.]

i. e. $\overline{bd+ace}^2 = a \times \overline{cd+be}^2 + 1.$

9. $\dfrac{\overline{bd+ace}^2 - 1}{\overline{cd+be}^2} = a.$

Prop.

Prop. 6.

But if $b^2 = acc + 1$, and $d^2 = aee - 1$, all other things remaining as in *Prop.* 3. Then shall $\overline{bd + ace}^2 + 1 = a \times \overline{cd + be}^2$. i. e. $\frac{\overline{bd + ace}^2 + 1}{\overline{cd + be}^2} = a$. which may be shewn, as the rest were.

Now, let a be any number proposed, and let the fraction $\frac{b}{c}$ be such, that either $\frac{bb - 1}{cc} = a$, or $\frac{bb + 1}{cc} = a$, and take the fractions $\frac{b}{ac}$ and $\frac{c}{b}$, before described; then the series of fractions converging to $\sqrt{a}$ will be as follows:

$\left.\frac{c}{b}, \frac{b}{ac}\right\} \frac{b}{c} =$ the first term of the series.

$\frac{bb + ace}{2bc} = \frac{d}{e}$ the second term.

$\frac{bd + acc}{cd + be} = \frac{f}{g}$ the third term.

$\frac{bf + acg}{cf + bg} = \frac{h}{k}$ the fourth term.

&c. *in infin.*

Every term is formed from the preceding; and the 2 fractions $\frac{b}{ac}$ and $\frac{c}{b}$, in the same manner as the second from the first, and these fractions.

And from the foregoing propositions it follows,

1. That if $\frac{bb - 1}{cc} = a$, then every fraction of the series shall be such.

That if from the square of its numerator be taken an unit, the remainder, divided by the square of its denominator, shall be equal to a.

For, by *Prop.* 1. the fraction $\frac{d}{e}$ shall be such; and by *Prop.* 3. the next fraction $\frac{f}{g}$ shall likewise be such; and so all the following terms.

Example.

Example.

Let $a=2$; then the firſt fraction, *i. e.* that in the ſmalleſt numbers, $\frac{b}{c}$, that makes $\frac{bb-1}{cc}=2$, is when $b=3$, and $c=2$; ſo that

$\left.\frac{c}{b}.\ \frac{b}{ac}\right\}\frac{b}{c}$ are And the terms following the firſt $\frac{3}{2}$

$\left.\frac{2}{3}.\ \frac{3}{4}\right\}\frac{3}{2}$ are $\frac{17}{12}.\ \frac{99}{70}.\ \frac{577}{408}.\ \frac{3363}{2378}.$ &c.

2. But if $\frac{bb+1}{cc}=a$, *i. e.* if the firſt fraction $\frac{b}{c}$ of the ſeries have the ſquare of its numerator an unit leſs than acc, the multiple of the ſquare of its denominator by the number a; the ſecond term ſhall have the ſquare of its numerator an unit greater than the ſaid multiple of the ſquare of its denominator; and the third term ſhall have the ſaid ſquare an unit leſſer, and ſo on alternately.

For, by *Prop.* 2. the ſecond term $\frac{d}{e}$ ſhall be ſuch, that $\frac{dd-1}{ee}=a$: and therefore, by *Prop.* 4. the third term $\frac{f}{g}$ ſhall be ſuch, that $\frac{ff+1}{cc}=a$. And by *Prop.* 5. it follows, that the next term $\frac{h}{k}$ ſhall be ſuch, that $\frac{hh-1}{kk}=a$; and ſo on alternately, by *Prop.* 4. and 5.

Example.

Let $a=2$; then the firſt fraction $\frac{b}{c}$ that makes $\frac{bb+1}{cc}=2$, is when $b=1$, and $c=1$. So that

$\frac{c}{d}$

$\frac{c}{b}.\ \frac{ac}{b}\Big\}\ \frac{b}{c}$

And the following terms

are $\Big\}\ \frac{1}{1}$ are $\frac{3}{2}.\ \frac{7}{5}.\ \frac{17}{12}.\ \frac{41}{29}.\ \frac{99}{70}$ &c.
$\frac{1}{1}.\ \frac{1}{2}$

But if a be 13, then the firſt fraction will be $\frac{18}{1}$
$\frac{5}{18}.\ \frac{18}{65}\Big\}\ \frac{18}{5}.\ \frac{649}{180}.\ \frac{23382}{6485}$ &c.

3. But if the fraction $\frac{b}{c}$ be ſuch, that $\frac{bb-1}{cc}=a$, and if the fractions $\frac{b}{ac}$, $\frac{c}{b}$, be taken, from which the ſeries is to be formed, as has been deſcribed; then, if the firſt fraction of the ſeries be made not $\frac{b}{c}$, but ſome fraction $\frac{d}{e}$, ſuch that $\frac{dd+1}{ee}=a$; then ſhall every term of the ſeries be ſuch as the fraction $\frac{d}{e}$, *i. e.* the ſquare of the numerator being increaſed by an unit, and the ſum divided by the ſquare of the denominator, the quotient ſhall be equal to a.

For, ſince $bb=acc+1$, and $dd=aee-1$, by *Prop.* 6. it follows, that the next term $\frac{f}{g}$ ſhall be ſuch, that $\frac{ff+1}{gg}=a$; and ſo on for every term.

Example.

Let $a=2$ $\frac{b}{c}=\frac{3}{2}$; then will $\frac{b}{ac}=\frac{3}{4}$, and $\frac{c}{b}=\frac{2}{3}$, and let $\frac{d}{e}=\frac{1}{1}$; then

$\frac{c}{b}.\ \frac{b}{ac}\Big\}\ \frac{d}{e}$

are

$\frac{2}{3}.\ \frac{3}{4}\Big\}\ \frac{1}{1}$

And

And the other terms

are $\frac{7}{5}$. $\frac{41}{29}$. $\frac{239}{169}$. $\frac{1393}{985}$. &c.

To find $\frac{b}{c}$ such as makes $bb - 1 = acc$, *i. e.* $acc + 1 = bb$, recourse must be had to Lord Brouncker's method in Dr. Wallis's *Commercium Epistolicum.*

LVII. *Observations upon the* Electricity *of the* Air, *made at the* Chateau de Maintenon, *during the Months of* June, July, *and* October, 1753; *being Part of a Letter from the Abbé* Mazeas, *F.R.S. to the Rev.* Stephen Hales, *D. D. F. R. S. Translated from the* French *by* James Parsons, *M.D. F. R. S.*

SIR,

Read Dec. 20, 1753.

BEING assured, that the electricity of the atmosphere would yet afford means of entertaining you, I spent part of this summer in observing what nature presented me upon so important a subject.

On the 14th of June I accompanied the Marechal de Noailles to his castle of Maintenon. At my arrival, I set up an apparatus, which consisted of an iron wire 370 feet long, raised to 90 feet above the horizon. It came down from a very high room in the castle, where it was fastened to a silken cord six